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SUBORDINATION ASSOCIATED WITH LAGUERRE POLYNOMIAL

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ABSTRACT. In this work, we have considered the Laguerre polynomial. This polynomial has been studied in several branches of theoretical physics and applied Mathematics. J. K. Prajapat at.al derived condition so that Laguerre polynomial satisfy convexity, strong starlikeness, close-to-convexity and strongly convexity. In this article, characteristics properties such as exponential subordination have been studied. Moreover Janowski starlikeness and convexity have been investigated for this polynomial. Several examples and corollaries have been mentioned to validates the result.

1. Introduction and Motivation

Let $\mathcal{A}$ denote the normalized collection of analytic functions, which satisfy the condition $F'(0) - 1 = F(0) = 0$. A function $F \in \mathcal{A}$ which is univalent in the unit disk such that $U = \frac{zF'}{F}$ or $U = 1 + \frac{zF''}{F'}$ lies in the domain $|\log U| < 1$ of the right half plane connected with the exponential function. Subordination play an important character in defining these geometric properties. If g and h are two analytic function in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, then g is subordinate to h if there exists a Schwartz function l in $\mathbb{D}$ such that $g = hol$ in $\mathbb{D}$. It can be denoted as $g \prec h$ if $g(0) = h(0)$ and $g(D) \subseteq h(D)$. The Class $\mathcal{P}_e$ contains analytic function p in $\mathbb{D}$ with $p(z) \prec e^z$ and $p(0) = 1$ for every $z \in \mathbb{D}$. A function $F \in \mathcal{A}$ is called exponential convex (or starlike) if $1 + \frac{zF''(z)}{F'(z)}$ or $\frac{zF'(z)}{F(z)}$ belong to $\mathcal{P}_e$. It is denoted as $\mathcal{K}_e$ and $\mathcal{K}_e^*$. Ma and Minda [14] introduced particular cases of subclass of starlike S^* and convex function $\mathcal{K}^*$. Mathematical charecterization of S^* and $\mathcal{K}^*$ is defined as follows:

$$\mathcal{S}^* = \left\{ F \in \mathcal{A} : \Re\left(\frac{zF'(z)}{F(z)}\right) > 0 \quad \text{for all } z \in \mathbb{D} \right\},$$

$$\mathcal{K} = \left\{ F \in \mathcal{A} : \Re\left(1 + \frac{zF''(z)}{F'(z)}\right) > 0 \quad \forall\, z \in \mathbb{D} \right\}.$$

A function $F \in \mathcal{A}$ is lemniscate convex if $1 + \frac{zF''(z)}{F'(z)}$ lie in bounded region by right half of lemniscate of Bernouli $\{k : |k^2 - 1 = 1\}$. In terms of subordination lemniscate starlike and lemniscate convexity is defined as $\frac{zF'(z)}{F(z)} \prec \sqrt{1+z}$ and $1 + \frac{zF''(z)}{F(z)} \prec \sqrt{1+z}$ respectively. For $-1 \le D < C \le 1$, suppose that $P[C, D]$ be the class containing normalized analytic function $p(z) = 1 + c_1(z) + \cdots$ in $\mathbb{D}$ satisfying $p(z) \prec \frac{1+Cz}{1+Dz}$. For $0 \le \beta < 1$, then $P[1 - 2\beta, -1]$ is the class of function $1 + c_1(z) + \cdots$ holds $\Re(p(z)) > \beta$ in $\mathbb{D}$. The class $S^*[C, D]$ of Janowski Starlike function [1] if $F \in \mathcal{A}$ satisfy $\frac{zF'(z)}{F(z)} \in P[C, D]$.

The extended Laguerre polynomials or Sonine polynomials of degree n with parameter $\alpha \in \mathbb{R}$ are given by Rodrigues formulae

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha} e^z}{n!} \left(\frac{d}{dz}\right)^n (z^{n+\alpha} e^{-z}).$$

They hold the condintion of secnd order differential equation,

$$zL'' + (\alpha + 1 - z)L'(z) + nL(z) = 0. \tag{1.1}$$

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Particularly for $\alpha = 0$ is the Laguerre polynomial $L_n(z)$ which has been discussed in literature. For $-1 < \alpha$ and $n \in N \cup 0$, the extended Laguerre polynomial $L_n(\alpha)(z)$ can be expressed as

$$L_n(\alpha)(z) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha}{n-k} \frac{z^k}{k!} \tag{1.2}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k (1+\alpha)_n}{(1+\alpha)_k (n-k)!} \frac{z^k}{k!}; \tag{1.3}$$

where $(z)_n = \frac{\Gamma(z+n)}{\Gamma(z)}$, $n \in \mathbb{N} \cup 0$ is a pochhamer symbol.

The Laguerre polynomials are vital in applied and theoretical physics as well as Mathematics. They come in quantum mechanics, in the radial part of the solution of the schrodinger equation of one electron atom. They represent the static winger function of oscillator systems and quantum mechanics too. The irreducibility of Laguerre polynomials, one may refer to (cited there in).

The analytical investigation of the Laguerre polynomials have been broadly done. Never the less its behaviour as an analytic functions has not explored much. The geometric behaviour of special functions were explored several researcher such discussions are apperared for the bessel function [3, 4], hypergeometric function [7, 9], Wright function [2], Mittag-Leffler function [5, 6], Coulumb wave function [13], incomplete beta function [8], stuve and Lommel function [2].

For $\alpha > -1$ and $n \in \mathbb{N} \cup 0$, the normalize Laguerre polynomials are defined by

$$M_{n,\alpha}(z) = \frac{n!}{(1+\alpha)_n} L_n^{\alpha}(z) = \sum_{k=0}^{n} \frac{(-1)^k n!}{(1+\alpha)_k (n-k)!} \frac{z^k}{k!},$$

Which hold differential equation too. If we put $n = 0, 1, 2, 3, \dots$ we can get the some terms of $M_{n,\alpha}$;

$$\begin{aligned}
M_{0,\alpha}(z) &= 1, M_{1,\alpha}(z) = 1 - \frac{z}{1+\alpha} \\
M_{2,\alpha}(z) &= 1 - \frac{2z}{1+\alpha} + \frac{z^2}{(1+\alpha)(2+\alpha)} \\
M_{3,\alpha} &= 1 - \frac{3z}{1+\alpha} + \frac{3z^2}{(1+\alpha)(2+\alpha)} - \frac{z^3}{(1+\alpha)(2+\alpha)(3+\alpha)}.
\end{aligned}$$

This polynomial also related to special functions like special functions like special function $J_\alpha(z)$, Whitteker function $W_{a,b}(z)$, hypergeometric function and Hermite polynomial.

$$\begin{aligned}
M_{n,\alpha}(z) &= \phi(-n; \alpha+1; z) \\
M_{n,\alpha}(z) &= z^{\frac{-1}{2}(\alpha+1)} e^{\frac{z}{2}} W_{n+\frac{1}{2},\frac{\alpha}{2}}(z), \\
\sum_{0}^{\infty} M_{n,\alpha}(z) \frac{t^n}{n!} &= \Gamma(1+\alpha)(zt)^{\frac{-\alpha}{2} e^t J_\alpha(2\sqrt{z}t)} \\
M_{n,-1/2}(z^2) &= \frac{(-1)^n n!}{2n!} H_{2n}(z).
\end{aligned}$$

In addition, we can find that

$$M'_{n,\alpha}(z) = -\frac{n}{(1+\alpha)} M_{n-1,\alpha+1}(z).$$

In order to show the main findings, we need lemmas, which are as follows:

2. Useful Lemmas

Some Lemmas have been recollected in the below section, which will be useful to show our main results.

Lemma 1. [14] Let Ω be a subset of $\mathbb{C}$ and the function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ holds the condition $\psi(r, s, t : z) \notin \Omega$ whenever: $s = me^{i\theta}e^{e^{i\theta}}, r = e^{e^{i\theta}}$, $\Re((s+t)e^{-i\theta}e^{-e^{i\theta}}) \geq 0$, $\theta \in [0, 2\pi), z \in \mathbb{D}$ and $1 \leq m$. If q is an analytic function in $\mathbb{D}$ with $q(0) = 1$ and $\psi(q(z), zq'(z), z^2q''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $q \in \mathcal{P}_e$.

It can be noted that the admissibility condition $\psi(r, s, t; z) \notin \Omega$ is verified for all : $s = me^{i\theta}e^{e^{i\theta}}$, $r = e^{e^{i\theta}}$, $\Re((s+t)e^{-i\theta}e^{-e^{i\theta}} \geq 0, \theta \in [0, 2\pi), z \in \mathbb{D}$ and $1 \leq m$. Furthermore for $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$ the admissibility condition convert into $\psi(e^{e^{i\theta}}, me^{i\theta}e^{e^{i\theta}}) \notin \Omega$, where $z \in \mathbb{D}$, $\theta \in [0, 2\pi)$ and $m \geq 1$.

Lemma 2. [15] Suppose that $\Omega \subset \mathbb{C}$ and $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ hold $\psi(i\rho, \sigma, \mu + iv; z) \notin \Omega$ when $z \in \mathbb{D}$, ρ real $\sigma \leq -\frac{1+\rho^2}{2}$ and $\sigma + \mu \leq 0$. If q is analytic in $\mathbb{D}$ with $q(0) = 1$ and $\psi(q(z), zq'(z), z^2q''(z); z) \in \Omega$ for $z \in \mathbb{D}$, then $\Re(q(z)) > 0$ in $\mathbb{D}$.

In this case $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$, then the condition Lemma 2 convert into
$\psi(i\rho, \sigma; z) \notin \Omega$ for ρ real and $\sigma \leq -\frac{1+\rho^2}{2}$.

Lemma 3. [12] Assume q in normalized class of analytic function with $q(z) \neq 1$. Let $\Omega \in \mathbb{C}$ and $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ hold $\psi(i\rho, \sigma, \mu + iv; z) \notin \Omega$ when $z \in \mathbb{D}$, $r = \sqrt{2\cos 2\theta}e^{i\theta}$, $s = \frac{me^{3i\theta}}{2\sqrt{2\cos 2\theta}}$, $\Re((s+t)e^{-3i\theta}) \geq \frac{3m^2}{8\sqrt{2\cos 2\theta}}$, $m \geq n \geq 1$ and $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$. If $(q(z), zq'(z), z^2q''(z); z) \in \mathbb{D}$ for $z \in \mathbb{D}$ and $\psi(q(z), zq'(z), z^2q''(z); z) \in \Omega$, $z \in \mathbb{D}$ then $q(z) \prec \sqrt{1+z}$.

In this case $\psi : \mathbb{C}^2 \times \mathbb{D} \to \mathbb{C}$, the condition in Lemma 3, reduces to $\psi(r, s; z)$ whenever $r = \sqrt{2\cos 2\theta}e^{i\theta}$, $s = \frac{me^{3i\theta}}{2\sqrt{2\cos 2\theta}}$, $m \geq n \geq 1$ and $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$ and $z \in \mathbb{D}$.

3. Exponential Subordination

In this section main results related to exponential subordination involving Laguerre polynomial has been investigated.

Theorem 1. *Assume that $\alpha > -1$, $n \in \mathbb{N} \cup 0$ and hold the condition $\frac{1}{e}[\Re(\alpha - 1) - n] > 0$, then $M_{n,\alpha}(z) \in \mathcal{P}_e$.*

Proof. Let $q : \mathbb{D} \to \mathbb{C}$ be given by $q(z) = M_{n,\alpha}(z)$. Here $q(z)$ is holomorphic function with $q(0) = 1$. By equation 1.1, $q(z)$ hold the second order differential equation, we get

$$z^2q''(z) + (\alpha + 1 - z)zq'(z) + znq(z) = 0.$$

Let us consider an another function $\psi : \mathbb{C}^3 \times \mathbb{D} \to \mathbb{C}$ by

$$\psi(r, s, t : z) = t + (\alpha + 1 - z)s + rnz.$$

Assume that $\Delta = 0$. Then $\psi(q(z), zq'(z), z^2q'' : z) \in \Delta$, $\forall z \in \mathbb{D}$.
Now, we prove that $q(z) \prec e^z$, using lemma 1. We have $s = me^{i\theta}e^{e^{i\theta}}, r = e^{e^{i\theta}}$, $\Re((s+t)e^{-i\theta}e^{-e^{i\theta}}) \geq 0$, $\theta \in [0, 2\pi), z \in \mathbb{D}$ and $1 \leq m$. Remind triangle inequality $|z_1 - z_2| \geq ||z_1| - |z_2||$, for all $z_1, z_2 \in \mathbb{C}$ and take into account

$$\begin{aligned}
|\psi(r, s, t; z)| &= |e^{e^{i\theta}}||(t+s)e^{-e^{i\theta}} + (\alpha - z)me^{e^{i\theta}} + nz| \\
&\geq e^{\cos\theta}||(t+s)e^{-i\theta}e^{-e^{i\theta}} + (\alpha - z)m| - |nz|| \\
&> \frac{1}{e}[\Re(\alpha - 1) - n].
\end{aligned}$$

By given hypothesis $\frac{1}{e}[\Re(\alpha - 1) - n] > 0$. We get the desired result. $\square$

Remark 1. *After setting particular value of α, n and by theorem 1, we obtain following examples which are subordinate to exponential function e^z:*

Example 1. $M_{1,3}(z) = 1 - \frac{z}{4}$

Example 2. $M_{2,4}(z) = 1 - \frac{2z}{5} + \frac{z^2}{40}$

Example 3. $M_{3,5} = 1 - \frac{3z}{6} + \frac{3z^2}{42} - \frac{z^3}{336}$

Example 4. *Hypergeometric function* $M_{2,4} = \phi(-2, 5; z)$.

Example 5. *Wright function* $M_{3,5} = z^{-3} e^{\frac{z}{2}} W_{6,\frac{5}{2}}(z)$

Example 6. *Hermite polynomial* $M_{4,\frac{-5}{2}}(z^2) = \frac{H_8(z)}{8.7.6.5}$.

In the next section, we have obtained sufficient conditions such that Laguerre polynomial belongs to Janowski convexity. Further we will show that $M_{n,\alpha}(z)$ belong to Janowski starlikeness $S^*[C, D]$.

4. Janowski starlikeness and convexity of Laguerre polynomial

Theorem 2. *Let $\alpha > -1, n \in \mathbb{N}$,*

$$h_1 = \Big[- (C-D)(1+D) - (n+1)(1+D)^2 \Big] \Big[- (C-D)(1-D) + (n+1)(1-D)^2 \Big] > 0,$$

$$\begin{aligned} h_2 = {} & \Big[(C-D) + (C-D)^2 + (C-D)(1+D)(\alpha+1) \Big] \Big[- (C-D)(1-D) + (n+1)(1-D)^2 \Big] \\ & + \Big[- (C-D)(1+D) - (n+1)(1+D)^2 \Big] \Big[(C-D) - (C-D)^2 + (C-D)(1-D)(\alpha+1) \Big], \end{aligned}$$

$$h_3 = \Big[(C-D) + (C-D)^2 + (C-D)(1+D)(\alpha+1) \Big] \Big[(C-D) - (C-D)^2 + (C-D)(1-D)(\alpha+1) \Big].$$

and $-1 \le D < C \le 1$. Whenever

$$(C-D)(1+D) + (n+1)(1+D)^2 > 0,$$

inequality hold

$$2(C-D) + 2(C-D)^2 + (C-D)(1+D)\alpha - (n+1)(1+D)^2 > 0. \tag{4.1}$$

Whenever

$$(C-D)(1+D) + (n+1)(1+D)^2 < 0,$$

inequality hold

$$2(C-D) + 2(C-D)^2 + (C-D)(1+D)(\alpha+2) + (n+1)(1+D)^2 > 0. \tag{4.2}$$

Whenever $|\frac{-h_2}{2h_1}| < 1$, inequality hold

$$\begin{aligned} & \max \Big\{ \big[- (C-D)^2 - D(C-D)(\alpha+2) + (n+1)(1-D^2) \big]^2, \\ & \qquad \big[(C-D)^2 + D(C-D)\alpha + (n+1)(1-D^2) \big]^2 \Big\} \\ & < h_3 - \frac{h_2^2}{4h_1}. \end{aligned}$$

Whenever $2h_1 + h_2 \le 1$, inequality satisfy

$$\begin{aligned} & \max \Big\{ \big[- (C-D)^2 - D(C-D)(\alpha+2) + (n+1)(1-D^2) \big]^2, \\ & \qquad \big[(C-D)^2 + D(C-D)\alpha + (n+1)(1-D^2) \big]^2 \Big\} \\ & < h_1 + h_2 + h_3. \end{aligned}$$

If $0 \notin M'_{n,\alpha}(\mathbb{D})$ *and* $0 \notin M''_{n,\alpha}(\mathbb{D})$, *then*

$$1 + z\frac{M''_{n,\alpha}(z)}{M'_{n,\alpha}(z)} \prec \frac{1+Cz}{1-Dz}.$$

Proof. Define $q : \mathbb{D} \to \mathbb{C}$

$$q(z) = \frac{(C-D)\phi'(z) + (1-D)z\phi''(z)}{(C-D)\phi'(z) - (1+D)z\phi''(z)},$$

where $\phi(z) = M_{n,\alpha}(z)$.

Then

$$q(z)\Big((C-D)\phi'(z) - (1+D)z\phi''(z)\Big) = (C-D)\phi'(z) + (1-D)z\phi''(z),$$

which gives

$$(C-D)\phi'(z)(q-1) = z\phi''(z)\Big((q+1) + D(q-1)\Big).$$

Thus

$$\frac{z\phi''(z)}{\phi'(z)} = \frac{(C-D)(q(z)-1)}{(q(z)+1) + D(q(z)-1)}.$$

Taking logarithmic derivative,

$$\frac{d}{dz}\log\left(\frac{z\phi''}{\phi'}\right) = \frac{d}{dz}\log\left(\frac{(C-D)(q-1)}{(q+1)+D(q-1)}\right). \tag{4.3}$$

Left hand side becomes

$$\frac{1}{z} + \frac{\phi'''(z)}{\phi''(z)} - \frac{\phi''(z)}{\phi'(z)}.$$

Right hand side becomes

$$\frac{q'(z)}{q(z)-1} - \frac{(D+1)q'(z)}{(q(z)+1) + D(q(z)-1)}.$$

Thus

$$\frac{1}{z} + \frac{\phi'''}{\phi''} - \frac{\phi''}{\phi'} = \frac{q'(z)}{q(z)-1} - \frac{(D+1)q'(z)}{(q(z)+1) + D(q(z)-1)}. \tag{4.4}$$

Multiplying by z both sides,

$$1 + \frac{z\phi'''}{\phi''} - \frac{z\phi''}{\phi'} = \frac{zq'(z)}{q(z)-1} - \frac{(D+1)zq'(z)}{(q(z)+1) + D(q(z)-1)}. \tag{4.5}$$

Hence

$$\frac{z\phi'''}{\phi''} = \frac{2zq'(z)}{(q(z)-1)((q(z)+1) + D(q(z)-1))} \; -1 + \frac{z\phi''}{\phi'}.$$

Multiplying,

$$\frac{z\phi'''(z)}{\phi''(z)}\frac{z\phi''(z)}{\phi'(z)} = \frac{2(C-D)zq'(z)}{((q(z)+1) + D(q(z)-1))^2} - \frac{(C-D)(q(z)-1)}{(q(z)+1) + D(q(z)-1)} + \frac{(C-D)^2(q(z)-1)^2}{((q(z)+1) + D(q(z)-1))^2}.$$

Now consider differential equation,

$$z\phi''(z) + (\alpha + 1 - z)\phi'(z) - n\phi(z) = 0.$$

Differentiating,

$$\phi''(z) + z\phi'''(z) - \phi'(z) + (\alpha + 1 - z)\phi''(z) - n\phi'(z) = 0.$$

Thus

$$z\phi'''(z) + (\alpha + 2 - z)\phi''(z) - (n+1)\phi'(z) = 0.$$

Dividing by $\phi'(z)$,

$$z\frac{\phi'''(z)}{\phi'(z)}+(\alpha+2-z)\frac{\phi''(z)}{\phi'(z)}-(n+1)=0.$$

Multiplying by z,

$$z^2\frac{\phi'''(z)}{\phi'(z)}+(\alpha+2-z)z\frac{\phi''(z)}{\phi'(z)}-(n+1)z=0. \tag{4.6}$$

Using substitution,

$$\left(\frac{z\phi'''(z)}{\phi''(z)}\frac{z\phi''(z)}{\phi'(z)}\right)+(\alpha+2-z)\frac{z\phi''(z)}{\phi'(z)}-(n+1)z=0.$$

$$\begin{aligned}
&=\frac{2(C-D)zq'(z)}{\big((q(z)+1)+D(q(z)-1)\big)^2}-\frac{(C-D)(q(z)-1)}{(q(z)+1)+D(q(z)-1)}+\frac{(C-D)^2(q(z)-1)^2}{\big((q(z)+1)+D(q(z)-1)\big)^2}\\
&+(\alpha+2-z)\frac{(C-D)(q(z)-1)}{(q(z)+1)+D(q(z)-1)}-(n+1)z\\
&=\frac{2(C-D)zq'(z)}{\big((q(z)+1)+D(q(z)-1)\big)^2}+\frac{(C-D)^2(q(z)-1)^2}{\big((q(z)+1)+D(q(z)-1)\big)^2}+\frac{(C-D)(q(z)-1)(\alpha+1-z)}{(q(z)+1)+D(q(z)-1)}\\
&\quad-(n+1)z=0.
\end{aligned}$$

Let us simplify

$$A=(q(z)+1)+D(q(z)-1)=(1+D)q+(1-D).$$

Multiplying throughout by A^2, we obtain

$$\begin{aligned}
&2(C-D)zq'+(C-D)^2(q-1)^2\\
&\quad+(C-D)(q-1)(\alpha+1-z)A-(n+1)zA^2=0.
\end{aligned} \tag{4.7}$$

Now expand each term.

$$(q-1)^2=q^2-2q+1.$$

$$\begin{aligned}
(q-1)A&=(q-1)\big((1+D)q+(1-D)\big)\\
&=(1+D)q^2-2Dq-(1-D).
\end{aligned}$$

$$\begin{aligned}
A^2&=((1+D)q+(1-D))^2\\
&=(1+D)^2q^2+2(1-D^2)q+(1-D)^2.
\end{aligned}$$

Substituting these expansions and collecting like terms, we obtain

$$\begin{aligned}
&2(C-D)zq'(z)\\
&+\Big[(C-D)^2+(C-D)(1+D)(\alpha+1-z)-(n+1)z(1+D)^2\Big]q^2(z)\\
&+\Big[-2(C-D)^2-2D(C-D)(\alpha+1-z)-2(n+1)z(1-D^2)\Big]q(z)\\
&+\Big[(C-D)^2-(C-D)(1-D)(\alpha+1-z)-(n+1)z(1-D)^2\Big]=0.
\end{aligned} \tag{4.8}$$

Hence, the equation takes the form

$$F_1zq'(z)+F_2q(z)^2+F_3q(z)+F_4=0,$$

where

$$\begin{aligned}
&F_1 = 2(C-D),\\
&F_2 = (C-D)^2 + (C-D)(1+D)(\alpha+1-z) - (n+1)z(1+D)^2,\\
&F_3 = -2(C-D)^2 - 2D(C-D)(\alpha+1-z) - 2(n+1)z(1-D^2),\\
&F_4 = (C-D)^2 - (C-D)(1-D)(\alpha+1-z) - (n+1)z(1-D)^2.
\end{aligned}$$

With the help of (4.8), $\Delta = 0$ imply that $\psi(q(z), zq'(z); z) \in \Delta$. Assume that $G_1 = \Re(F_1) = 2(C-D)$, $G_2 =$

$$\Re(F_2) = (C-D)^2 + (C-D)(1+D)(\alpha+1-x) - (n+1)x(1+D)^2,$$

$$G_3 = \Re(iF_3) = -2(C-D)^2 - 2D(C-D)(\alpha+1+y) + 2(n+1)y(1-D^2),$$

$$G_4 = (C-D)^2 - (C-D)(1-D)(\alpha+1-x) - (n+1)x(1-D)^2.$$

For $\sigma \leq -\frac{(1+\rho^2)}{2}$, $\rho \in \mathbb{R}$.

$$\begin{aligned}
&\Re\psi(i\rho, \sigma; z) = G_1\sigma + G_2(i\rho)^2 + G_3\rho + G_4\\
&= G_1\sigma - G_2(\rho)^2 + G_3\rho + G_4\\
&\leq G_1\left(\frac{-(1+\rho^2)}{2}\right) - G_2\rho^2 + G_3\rho + G_4\\
&\leq \frac{-G_1}{2} - \frac{G_1\rho^2}{2} - G_2\rho^2 + G_3\rho + G_4\\
&= \frac{-G_1 - G_1\rho^2 - 2G_2\rho^2 + 2G_3\rho + 2G4}{2}\\
&= \frac{-1}{2}[(G_1+2G_2)\rho^2 - 2G_3\rho + G_1 - 2G_4] = Q(\rho).
\end{aligned}$$

By given hypothesis from any of (4.1) and (4.2) $(G_1 + 2G_2) > 0$, $Q(\rho)$ has maximum value at $\rho = \frac{G_3}{G_1+2G_2}$. Now finding value of $Q(\rho)$ at $\rho = \frac{G_3}{G_1+2G_2}$, for all $\rho, |x|, |y| < 1$, we have

$$\begin{aligned}
&Q(\rho) = \frac{-1}{2}[(G_1+2G_2)\left(\frac{G_3}{G_1+2G_2}\right)^2 - 2G_3\left(\frac{G_3}{G_1+2G2}\right) - 2G_4 + G_1] < 0\\
&= \frac{-1}{2}\left[\frac{-(G_3)^2}{G_1+2G_2} - 2G_4 + G_1\right] < 0\\
&= \frac{G_3^2}{G_1+2G_2} < G_1 - 2G_4\\
&= G_3^2 < (G_1+2G_2)(G_1-2G_4).
\end{aligned}$$

For $|x| < 1, |y| < 1$ and $y^2 < 1 - x^2$ for above inequality, we have to prove

$$\begin{aligned}
&[-(C-D)^2 - D(C-D)(\alpha+1+y) + (n+1)y(1-D^2)\Big]^2\\
&< \Big[(C-D) + (C-D)^2 + (C-D)(1+D)(\alpha+1-x) - (n+1)x(1+D)^2\Big]\\
&\times \Big[(C-D) - (C-D)^2 + (C-D)(1-D)(\alpha+1-x) + (n+1)x(1-D)^2\Big]. \qquad (4.9)
\end{aligned}$$

Now, Verification of the required inequality,

$$L(y) = -(C-D)^2 - D(C-D)(\alpha+1+y) + (n+1)y(1-D^2).$$

Let $K = C - D$. Then

$$L(y) = -K^2 - DK(\alpha + 1 + y) + (n+1)y(1 - D^2)$$
$$= -\big[K^2 + DK(\alpha+1)\big] + y\big[(n+1)(1-D^2) - DK\big].$$

Define

$$A = K^2 + DK(\alpha+1), \quad B = (n+1)(1-D^2) - DK.$$

Thus,

$$L(y) = -A + By.$$

Determine the maximum value of left hand side . Since $y^2 < 1 - x^2 \le 1$, we have $y \in (-1, 1)$. Hence,

$$\max_{|y|<1} L(y)^2 = \max\{(-A+B)^2, (A+B)^2\}.$$

That is,

$$\max L(y)^2 = \max\Big\{\big[-K^2 - DK(\alpha+2) + (n+1)(1-D^2)\big]^2, \big[K^2 + DK\alpha + (n+1)(1-D^2)\big]^2\Big\}.$$

Now, we determine the minimum value of right hand side
Let

$$H(x) = \Big[(C-D) + (C-D)^2 + (C-D)(1+D)(\alpha+1-x) - (n+1)x(1+D)^2\Big] \times \Big[(C-D) - (C-D)^2 + (C-D)(1-D)(\alpha+1-x) + (n+1)x(1-D)^2\Big].$$

Writing $K = C - D$, we express

$$H(x) = h_1x^2 + h_2x + h_3,$$

where

$$h_1 = \Big[-(C-D)(1+D) - (n+1)(1+D)^2\Big]\Big[-(C-D)(1-D) + (n+1)(1-D)^2\Big],$$

$$h_2 = \Big[(C-D) + (C-D)^2 + (C-D)(1+D)(\alpha+1)\Big]\Big[-(C-D)(1-D) + (n+1)(1-D)^2\Big] + \Big[-(C-D)(1+D) - (n+1)(1+D)^2\Big]\Big[(C-D) - (C-D)^2 + (C-D)(1-D)(\alpha+1)\Big],$$

$$h_3 = \Big[(C-D) + (C-D)^2 + (C-D)(1+D)(\alpha+1)\Big]\Big[(C-D) - (C-D)^2 + (C-D)(1-D)(\alpha+1)\Big].$$

By given condition $h_1 > 0$, then the minimum occurs at

$$\nu = -\frac{h_2}{2h_1},$$

since $H'(x) = 0$ at $-\frac{h_2}{2h_1}$ and $H''(x) = 2 > 0$. Therefore minimum of H(x) will be $h_3 - \dfrac{h_2^2}{4h_1}$ for $\nu \in (-1, 1)$. In this case $|\nu| \ge 1$ and $H'(x) = 2h_1x + h_2 \le 2h_1 + h_2 \le 0$ equivalent to given condition, $H(x)$ is decreasing. Hence $H(x) \ge H(1) = h_1 + h_2 + h_3 \ge 0$. Thus,

$$\min_{|x|<1} H(x) = \begin{cases} h_3 - \dfrac{h_2^2}{4h_1}, & \nu \in (-1, 1), \\ h_1 + h_2 + h_3, & \text{otherwise.} \end{cases}$$

We obtain equivalent to given condition of the theorem ,

$$\max\Big\{\big[-(C-D)^2-(C-D)K(\alpha+2)+(n+1)(1-D^2)\big]^2,$$
$$\big[(C-D)^2+D(C-D)\alpha+(n+1)(1-D^2)\big]^2\Big\}$$
$$\leq \min_{|x|<1} H(x).$$

Which proves the required result.

□

Corollary 1. *Let us consider the relation, we have*

$$zM_{n,\alpha}(z)=-\frac{\alpha}{n+1}zM'_{n+1,\alpha-1}(z),$$

and consequently

$$z\frac{(zM_{n,\alpha}(z))'}{zM_{n,\alpha}(z)}=1+z\frac{M''_{n+1,\alpha-1}(z)}{M'_{n+1,\alpha-1}(z)}$$

Simultaneosly with the Theorem 2 and substituting $n=n-1,\alpha=\alpha+1$*, yields the following reuslt as* $zM_{n,\alpha}(z)\in S^*[C,D]$ *Janowski starlike.*

5. Conclusion

This work establishes new geometric properties of the Laguerre polynomial through the framework of exponential subordination and Janowski function theory. Sufficient conditions have been obtained for Janowski starlikeness and Janowski convexity, extending and complementing earlier results on convexity, close-to-convexity, strong starlikeness, and strong convexity. The derived results provide a broader understanding of the geometric behavior of Laguerre polynomials and strengthen their connection with geometric function theory. The included examples and corollaries illustrate the applicability and validity of the main results. It is expected that these findings may motivate further investigations on other special functions and related subclasses defined via differential subordination techniques.

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